Latin trades on three or four rows

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Abstract

Latin trades are closely related to the problem of critical sets in Latin squares. We denote the cardinality of the smallest critical set in any Latin square of order n by $scs(n)$. A consideration of Latin trades which consist of just two columns, two rows, or two elements establishes that $scs(n) \geq n-1$. We conjecture that a consideration of Latin trades on four rows may establish that $scs(n) \geq 2n-4$. We look at various attempts to prove a conjecture of Cavenagh about such trades. The conjecture is proven computationally for values of n less than or equal to 9. In particular, we look at Latin squares based on the group table of $\mathbb{Z}_n$ for small n and trades in three consecutive rows of such Latin squares.

Key words: Latin squares, Latin trades, integer programming

1 Introduction

A *Latin square* L of order n is an $n \times n$ array of entries $\{(i,j;k)\}$, where i, j and k are from an n element set, such that each row and column of L contains each of the n possible elements exactly once. In each cell $(i,j;k)$, i is the row number, j is the column number, and k is the element. We will refer to the Latin square based on the group table of $\mathbb{Z}_n$ simply as $\mathbb{Z}_n$.

Let L be a Latin square of order n and let $\{a,b,c\} = \{1,2,3\}$. The (a,b,c)-*conjugate* of L, written $L_{(a,b,c)}$, is defined as follows:

$$L_{(a,b,c)} = \{(x_a, x_b; x_c) \mid (x_1, x_2; x_3) \in L\}.$$

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Two Latin squares L and L' of order n are *isotopic* if there are three bijections from the rows, columns, and symbols of L to the rows, columns, and symbols, respectively, of L', that map L to L'. Two Latin squares L and L' of order n are *main class isotopic* if L is isotopic to any conjugate of L'.

A *uniquely completable (UC)* set U is a subset of a Latin square L such that L is the only superset of C which is a Latin square. A *critical set* C of L is a subset of L such that C is uniquely completable and no subset of C has this property. In this paper the number of entries in a critical set C will be referred to as its *size*, as in the titles of the papers [3], [5] and [11]. Critical sets were introduced by Nelder [17].

A closely related concept is that of the *Latin trade.* A Latin trade is the set of entries in which two Latin squares of the same order differ. It is said to be *minimal* if no subset is a Latin trade. The *disjoint mate* of a Latin trade is the "other" difference between two Latin squares L and M, that is, if a trade is $M \setminus L$, its disjoint mate is $L \setminus M$. The connection between uniquely completable sets and Latin trades is well-known, for example see [10]. It is expressed in the following lemma.

Lemma 1 *In any critical set C for a Latin square L, each trade in L must intersect C in at least one entry. Also, for each entry x in C, there must be a trade in L which intersects C only in the entry x.*

The function $scs(n)$ was defined by Nelder as the smallest cardinality of a critical set in any $n \times n$ Latin square. Independently, Nelder [18], Bate and van Rees [3], and Mahmoodian [16] conjectured that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$. It is known that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$ for $n \leq 8$ ([3],[7],[2],[4]). Ghandehari et al [12] introduced the abuse of notation scs(L) to refer to the cardinality of a minimal critical set in a Latin square L. We will use this notation later.

In fact, we know that $scs(n) \leq \lfloor \frac{n^2}{4} \rfloor$ from Curran and van Rees [7]. For a specific value of n, then, suppose we are able to find a representative of each main class of Latin squares of order n. Call such a representative Latin square L.

If we can find the set of all minimal Latin trades, $\mathcal{T}$, which are contained in L, and show that for any subset $S \subseteq L$ of cardinality $< \lfloor \frac{n^2}{4} \rfloor$ there exists a trade $T \in \mathcal{T}$ not intersecting S, this would show that S does not have unique completion to L, thus demonstrating that $scs(L) \geq \lfloor \frac{n^2}{4} \rfloor$. Proving this over all main classes of a given order n would establish that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$ for the given n.

However, even for relatively small n, it is computationally infeasible to examine all such subsets or to find the set of all minimal Latin trades $\mathcal{T}$.

Instead of examining all such subsets, or generating the set of all minimal Latin trades, we can use integer programming techniques to find the minimum size of a set in L which intersects every trade from a given set $\mathcal{T}_1$ of trades in L. We do this by writing an integer program, with the objective function minimizing the sum of a set of binary variables corresponding to cells, with each variable set to 0 if the cell is empty and 1 if the cell is non-empty. Each constraint of the integer program corresponds to a trade from $\mathcal{T}_1$ on less than or equal to three rows, columns or elements; we ensure that the sum of the variables corresponding to each cell in a trade is at least 1.

If this minimum size is at least $\lfloor \frac{n^2}{4} \rfloor$, this establishes that $scs(L) \geq \lfloor \frac{n^2}{4} \rfloor$, and if such an IP (integer program) is solved over all main classes of a given order n, with the solution in each case being at least $\lfloor \frac{n^2}{4} \rfloor$, this shows that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$ for that n.

It was discovered experimentally that by setting $\mathcal{T}_1$ to be the set of Latin trades with less than or equal to three rows, columns, or elements, we could show that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$ for $4 \leq n \leq 7$ using an integer program with constraints based on these trades.

We define the "use" of trades as follows. Trades are "used" in an integer program by adding them to the program as constraints. In all existing proofs giving a new lower bound for $scs(n)$, say $scs(n) \geq f(n)$, trades are "used" in one of two ways.

In the first way, a partial Latin square P with less than $f(n)$ filled cells is shown to have at least two completions, L_1 and L_2, which is effectively showing that the trade $L_1 \setminus L_2$ in L_1 (or $L_2 \setminus L_1$ in L_2) does not intersect P.

In the second way, a subset S of a Latin square L with less than $f(n)$ filled cells is shown to have at least two completions due to the existence of a trade $T \in L$ which does not intersect S.

The first bound for $scs(n)$, $scs(n) \geq n - 1$, given by Curran and van Rees in 1978 used trades on less than or equal to two rows, columns, or elements. Thus using these trades in an IP over all the main classes of LSs of order n for a given n would show that $scs(n) \geq n - 1$.

Similarly, the proofs by Donovan et al [8], Cooper et al [6], Fu et al [11], and Horak et al [13] use slightly more complicated trades, and the trades found in each paper could be used in an integer program to show that the corresponding bounds on $scs(n)$ are true for particular values of n.

A logical question follows: is it possible to pick trades from Latin squares, build an IP using these trades, and conjecture then prove a bound on $scs(n)$ for all n, or perhaps just $scs(L)$ for some Latin square L?

Considering the above results obtained using trades on less than or equal to three rows, columns, or elements, for order $n \leq 7$, the author made the following conjecture.

Conjecture 1 *The use of trades on two or three columns, rows, and elements suffices to prove that* $scs(n) = \lfloor \frac{n^2}{4} \rfloor$.

This conjecture was disproved in [4] where these trades were used in IPs in order to determine $scs(8)$. In three of the 283,657 main classes examined, it was found that Latin trades on less than or equal to three rows, columns, or elements did not suffice to show that the smallest critical set size for Latin squares from these classes was at least $\lfloor \frac{8^2}{4} \rfloor$. That is, using such trades in an IP, the objective function value was less than $\lfloor \frac{8^2}{4} \rfloor$ for these three main classes.

2 Cavenagh's conjecture

Cavenagh wrote a paper [5] in which an algorithm was given to create Latin trades on three rows from any Latin square. This algorithm was used to establish that if a critical set C of order n contained an empty row, column, or a missing element, then $|C| \geq 2n - 4$. He also made a conjecture that "in any critical set P, there are at most three columns (or rows or entries) with at most one element". We restate this conjecture in a form more apposite for this paper.

Conjecture 2 *Take a* $4 \times n$ *Latin rectangle* R *and remove one entry from each row. Call the resulting partial Latin rectangle a chopped Latin rectangle. Then a Latin trade exists which is a subset of the resulting chopped Latin rectangle.*

Cavenagh claimed that a proof of this result together with his own result would prove that $scs(n) \geq 2n - 4$. In fact if this problem were to be solved, it would be sufficient to prove $scs(n) \geq 2n - 4$ on its own, as it shows that in any critical set there are less than or equal to three rows with less than or equal to one entry (similar statements apply for columns and elements).

Hereafter instead of saying that a chopped Latin rectangle R contains a Latin trade T which is a subset of R, we will write that T is "in" R. We will also refer to chopped Latin rectangles on both three and four rows later.

Although we do not prove this conjecture in this paper, using integer programming, we were able to identify trades in three consecutive rows of $\mathbb{Z}_n$ which enable us to decide if there is a trade in a chopped rectangle based on such rows. This helps to prove Conjecture 2 in cases which have such rows as a subset.

We also use IPs to analyse trades found in chopped Latin rectangles on four rows for orders $4 \leq n \leq 9$. We can determine the minimum number of trades needed to establish the truth of the conjecture for a given order n, or the minimum different number of sizes of trades needed for the same purpose. The rationale behind this is to try to prove the conjecture using a "simple" set of trades which we can generate with a relatively simple algorithm such as that found in Cavenagh [5].

Unfortunately, little is known about the structure of trades with more than two rows, columns, or elements, that is, the simplest trades. It is hoped that the results in this paper will help to find a proof for Conjecture 2 and perhaps eventually help to develop trades to prove the conjecture that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$.

Given the disproof of Conjecture 1, it is no longer logical to look for Latin trades on a fixed number of rows, columns or elements. Instead, if Conjecture 2 is proven, the next step should be to find Latin trades on six rows, columns, or elements for $n \geq 6$ in order to prove that $scs(n) \geq 3n - 9$, and in general to find Latin trades on $2x$ rows, columns, or elements for $n \geq 2x$ in order to prove that $scs(n) \geq xn - x^2$, which would establish that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$, our desired goal.

We consider Conjecture 2 for small values of n.

2.1 $n = 4$

There are two non-isomorphic 4×4 Latin rectangles, which are both group tables: the group table for $\mathbb{Z}_4$ and the group table for $\mathbb{Z}_2^2$.

It is logical at this point to ask the following question.

Question. What is the minimum number of trades or different trade sizes needed to prove that a trade exists as a subset of any chopped 4×4 Latin rectangle?

We can determine the minimum number of trades or different trade sizes required by using integer programming techniques.

The above question leads to an integer program with 11 constraints. Solving the IP shows that we need at least four different sizes of trades: 4, 8, 10 and 12.

- For $\mathbb{Z}_4$, we need at least two "sizes" of trades to cover all solutions: 8 and 10. The reason is that sometimes there is only one trade in the chopped rectangle, which is of size 8 or 10. Taking into account other cases where there are more trades, we find that we need at least three different sizes of trades: 4, 8, and 10; or 7, 8 and 10. The number of different trades required is at least 44.

Below, we give examples of chopped Latin rectangles from $\mathbb{Z}_4$ with exactly one trade in the chopped rectangle. For values of n to follow, similar examples are given. The bold entries are the entries being removed to make the chopped rectangle.

1	2	3	4
2	1	**4**	3
3	4	2	1
4	3	**1**	2

	2		4
	1		3
	4		1
	3		2

Chopped rectangle with one trade of size 8

1	2	3	4
2	1	**4**	3
3	4	2	1
4	3	1	**2**

	2	3	4
	1		3
	4	2	1
	3	1	

Chopped rectangle with one trade of size 10

- For chopped rectangles derived $\mathbb{Z}_2^2$, there is always more than one trade. Again at least two "sizes" are required: 4 and 12. There are 12 intercalates, and 8 trades of size 12 corresponding to the 8 transversals in $\mathbb{Z}_2^2$. For each transversal, the trade of size 12 is $\mathbb{Z}_2^2$ minus the transversal. This is explained in Khodkar [15]. The number of different trades required is at least 18.

1	2	3	4
2	1	4	3
3	**4**	1	2
4	3	**2**	1

	1		3
	3		1

		3	4
		4	3

Chopped rectangle with two trades of size 4

1	2	3	4
2	1	4	**3**
3	**4**	1	2
4	3	**2**	1

	2	3	4
2	1	4	
3		1	2
4	3		1

Chopped rectangle with one trade of size 12

2.2 $n = 5$

There are 3 non-isomorphic 4×5 Latin rectangles, one derived from the group table for $\mathbb{Z}_5$ and the other two from the non-$\mathbb{Z}_5$ main class.

Solving an IP problem with 103 constraints indicates we need at least seven different sizes of trades. To use exactly seven different sizes, sizes 8,9,12,14, and 16 are necessary and the other two sizes can be any of 6,10; 7,10; 4,6; 10,11; 10,13; 4,10; or 6,7.

- For a chopped rectangle derived from $\mathbb{Z}_5$, trades of size 8 are sufficient. There are always at least three trades in the chopped rectangle: when there are exactly three, these can be of sizes 8,12 and 16 or 8, 8 and 16. In both cases, the trade of size 16 is the union of the other two trades.
- For the non-$\mathbb{Z}_5$ main class, we need at least sizes 9,12,14 and 16, because sometimes there is exactly one trade of size 12, 14 or 16, or exactly two trades each of size 9 in the chopped rectangle.

1	2	3	4	5
2	1	**4**	5	3
3	4	5	1	**2**
4	**5**	2	3	1

	2	3	4	5
2	1		5	3
3	4		1	
4		2	3	

Chopped rectangle with one trade of size 14

1	2	3	4	5
2	1	4	5	**3**
3	**4**	5	1	2
4	5	**2**	3	1

	2	3	4	5
2	1	4	5	
3		5	1	2
4	5		3	1

Chopped rectangle with one trade of size 16

1	2	3	**4**	5
2	1	4	5	3
3	4	**5**	1	2
4	**5**	2	3	1

1	2	3		
	1	4		3
3	4			2
4		2		1

Chopped rectangle with one trade of size 12

1	2	3	4	5
2	1	4	**5**	3
3	4	5	1	2
4	5	**2**	3	1

		3	4	
	1	4		3
	4		1	
			3	1

		3		5
	1	4		3
	4	5		
	5			1

Chopped rectangle with two trades of size 9

2.3 $n = 6$

There are 56 non-isomorphic 4×6 Latin rectangles. Each chopped Latin rectangle contains at least three trades. When there are exactly three trades in the chopped Latin rectangle, the trades can be all of size 12; of sizes 10, 12, and 18; or of sizes 10, 10 and 20 with no intersection between the two trades of size 10.

If the trades are of exactly one size, the only size possible is 12, with exactly three trades in the chopped Latin rectangle, as above.

2	3	1	6	4	**5**
3	**1**	2	5	6	4
5	6	4	1	2	**3**
6	**4**	5	3	1	2

		1		4	
3		2	5	6	
		4		2	
6		5	3	1	

2		1	6		
		2	5	6	
5			1	2	
6		5		1	

2		1	6	4	
3			5		
5		4	1	2	
6			3		

Chopped rectangle with three trades of size 12

Solving an IP problem with 3343 constraints shows that at least 6 sizes of trades are necessary: 4,6,8,9,10 and 12. This is the unique minimum solution.

2.4 $n = 7$

There are 1398 non-isomorphic 4×7 Latin rectangles. Each chop-ped Latin rectangle contains at least three trades. When the chopped rectangle contains exactly three trades, the trades can be of sizes 12, 12 and 20; or of sizes 14, 14 and 24. But they cannot be of sizes 12, 12 and 24 with null intersection between the two trades of size 10; that is, the pattern of 4×5 and 4×6 Latin rectangles is not repeated.

When there are exactly 4 trades in the chopped rectangle, the trades can all be of size 21. So a method must be found to generate trades of this size, or else a way must be found to show that a trade similar to this exists. This is the only "unique" size required.

Solving an IP problem with 34683 constraints shows that at least 8 sizes of trades are necessary: 4,8,9,10,12,14,18 and 21. This is the unique minimum solution.

1	2	3	4	5	6	7
2	3	6	7	4	**5**	1
6	7	4	1	**2**	3	5
7	**4**	5	3	1	2	6

		3	4	5	6	7
	3	6	7	4		1
6	7	4	1		3	5
7		5	3	1		6

	2	3	4	5	6	7
2	3	6	7	4		1
		4	1			5
7		5	3	1	2	6

	2	3	4		6	7
2	3	6	7			1
6	7	4	1		3	5
7		5	3		2	6

	2	3	4	5	6	7
2	3	6	7	4		1
6	7				3	
7		5	3	1	2	6

Chopped rectangle, four trades of size 21

2.5 $n = 8$

There are 93561 non-isomorphic 4×8 Latin rectangles. Each chopped Latin rectangle contains at least seven trades.

There are at most three different sizes of trades in the chopped rectangle; if there are exactly 3, these sizes are 12,16,20; 4,8,24; or 12,18,24.

We can have 3 different chopped Latin rectangle derived from the same Latin rectangle for which the set of trades in the chopped rectangle is exactly the same in all three cases.

For some chopped Latin rectangles, trades containing at least 6 columns are required; when this is the case, there are at least 59 trades in the chopped rectangle.

Solving an IP problem with 175673 constraints shows that at least 7 different sizes of trades are necessary. There are 25 possible sets of size 7 containing different sizes.

When there are exactly 7 trades in the chopped rectangle, the trades can be of sizes found in Table 1 following.

Note that where two trades of size 28 occur, this is a so-called 3-way Latin trade ([1]).

2.6 $n = 9$

The minimum number of trades found in an examination of many 4×9 chopped Latin rectangles is seven, so it seems probable that there are at least seven trades in each chopped 4×9 Latin rectangle. (Similarly the minimum number of trades found in any 4×10 chopped Latin rectangle so far is 15.)

Given these results, we make the following conjecture.

Conjecture 3 *The minimum number of trades in any $n \times 4$ chopped Latin rectangle is $2^{\lfloor \frac{n}{2} \rfloor - 1} - 1$.*

Up to $n = 8$, in order to prove Conjecture 2, it was sufficient to derive all the non-isomorphic $4 \times n$ Latin rectangles, but for 4×9 there were too many. Therefore, the 2877 non-isomorphic 3×9 Latin rectangles were generated, and those where any chopped rectangle contained a trade were deleted. The remaining 422 3×9 rectangles had an extra row added to generate all the

Table 1
Possible sizes of trades remaining in 4×8 chopped Latin rectangle when there are exactly 7 trades remaining

4	10	10	12	16	22	22
4	10	10	24	24	28	28
6	12	16	16	18	24	24
6	16	16	16	16	22	28
8	12	14	16	20	22	24
8	12	14	18	20	22	26
8	12	18	18	20	26	26
8	14	14	16	16	24	28
8	14	14	18	18	28	28
8	18	18	18	18	24	28
10	10	12	18	20	22	28
10	12	12	18	20	22	26
10	12	14	16	20	20	24
12	12	12	16	18	22	24
12	12	12	20	20	24	28
12	12	16	16	16	20	20
12	12	16	18	24	24	26
12	16	16	18	18	24	28

non-isomorphic 4×9 Latin rectangles, which were tested by computer to demonstrate the truth of Conjecture 2 for $n = 9$. Thus $14 \leq scs(9) \leq 20$.

The examination of the different trades and trade sizes needed to prove Conjecture 2 seems to indicate that Latin trades with a complex structure are needed; in fact, these trades have a much more complicated structure than those trades for which we presently have constructions or algorithms. Unless a way is found to better classify these trades, it seems Conjecture 2 might be very hard to prove, except in specific cases.

2.7 Trades from $\mathbb{Z}_n$

We consider $\mathbb{Z}_n$. In any four rows of $\mathbb{Z}_n$, for $n > 4$, there must be three rows which are not isomorphic to three adjacent rows of $\mathbb{Z}_n$. (In a triple of rows isomorphic to three adjacent rows of $\mathbb{Z}_n$, each of the three pairs of rows in the triple is equidistant. Thus if there are four rows in $\mathbb{Z}_n$ in which all triples of rows have this property, this implies $n = 4$.) For $n \leq 36$ we looked at all possibilities for these rows by computer, and determined computationally that each chopped $3 \times n$ Latin rectangle contained a trade. Therefore, for $11 \leq n \leq 35$, n odd, we have $scs(\mathbb{Z}_n) \geq 2n - 4$. (Howse [14] showed that $scs(\mathbb{Z}_9) = 20$.)

2.8 Asymptotics of Conjecture 2

From Wilf [19] we know that as $n \to \infty$, the expected number of permutations in a cycle of length n also approaches ∞. So as $n \to \infty$, the probability of having a pair of rows in a random $4 \times n$ Latin rectangle with > 2 cycles goes to 1. (Since a pair of rows in a Latin square can be thought of as a permutation of the elements in the rows, we can write about a cycle in a pair of rows.) When there is such a pair of rows, we can remove 1 entry from each row and find a trade, showing that any critical set from a Latin square containing these four rows has at least $2n-4$ entries. Similarly, for $x > 2$, as $n \to \infty$, the probability of having a pair of rows in a random $2x \times n$ Latin rectangle with $> x$ cycles goes to 1. With such a pair of rows, we can remove $x - 1$ entries from each row and find a trade, which means that any critical set from a Latin square containing these $2x$ rows has at least $xn - x^2$ entries. Thus for any given x, as $n \to \infty$, the proportion of critical sets in Latin squares of order n with size at least $xn - x^2$ goes to 1.

3 Trades on three rows

Since no solution to the problem on four rows is evident, we solve a sub-problem, by extending the ideas of Cavenagh about trades in three consecutive rows of $\mathbb{Z}_n$. With the following theorem, we considerably extend the current knowledge about the structure of trades in chopped Latin rectangles based on such rows.

Theorem 1 *Take a $3 \times n$ Latin rectangle L consisting of three consecutive rows from $\mathbb{Z}_n$. Let n be odd and $n \geq 9$, or n be even and $n \geq 12$.*

Consider the following trades (for brevity, only the positions are given). x ranges from 0 to $n-1$, and we assume the trades are in the first three rows of $\mathbb{Z}_n$.

For n odd:

$$\begin{aligned}T_{even}(x) = &\{(0, x+2y|1 \leq y \leq \frac{n+1}{2}\} \cup \\ &\{(1,x),(1,x+1)\} \cup \\ &\{(2, x+2y|0 \leq y \leq \frac{n-1}{2}\}\end{aligned}$$

$$\begin{aligned}T_{odd}(x) = &\{(0, x+2y+1|0 \leq y \leq \frac{n-1}{2}\} \cup \\ &\{(1,x),(1,x+1)\} \cup \\ &\{(2, x+2y+1|0 \leq y \leq \frac{n-1}{2}\}\end{aligned}$$

For n even, let

$$\begin{aligned}T_a(x) = &\{(0,m)|m \text{ is even}\} \cup \\ &\{(1,x),(1,x+1),(1,x+2),(1,x+3)\} \cup \\ &\{(2,m)|m \text{ is even}\}\end{aligned}$$

$$\begin{aligned}T_b(x) = &\{(0,m)|m \text{ is odd}\} \cup \\ &\{(1,x),(1,x+1),(1,x+2),(1,x+3)\} \cup \\ &\{(2,m)|m \text{ is odd}\}\end{aligned}$$

where the column number m ranges from 0 to $n-1$.

Then, when x is even:

$$\begin{aligned}T_{even}(x) = &T_a(x) \cup \{(0,x+1),(0,x+3),(2,x+1)\} \setminus \\ &\{(0,x+2)\} \\ T_{odd}(x) = &T_b(x) \cup \{(0,x+2),(2,x),(2,x+2)\} \setminus \{(2,x+1)\}\end{aligned}$$

and when x is odd:

$$T_{even}(x) = T_a(x) \cup \{(0, x+2), (2, x), (2, x+2)\} \setminus \{(2, x+1)\}$$
$$T_{odd}(x) = T_b(x) \cup \{(0, x+1), (0, x+3), (2, x+1)\} \setminus \{(0, x+2)\}$$

Then, apart from five exceptional cases, Cases A to E specified below, there is a trade in any chopped rectangle R derived from L, and if a trade exists, a trade from the above sets can be found in the chopped rectangle.

Case A. $R_a = L \setminus \{(0, x; x), (2, x-1; x+1)\}$.

Case B. $R_b = L \setminus \{(0, x; x), (1, x; x+1), (2, x+1; x+3)\}$.

Case C. $R_c = L \setminus \{(0, x; x), (1, x+1; x+2), (2, x+1; x+3)\}$.

Case D. $R_d = L \setminus \{(0, x; x), (1, x-1; x), (2, x-3; x-1)\}$.

Case E. $R_e = L \setminus \{(0, x; x), (1, x-2; x-1), (2, x-3; x-1)\}$.

Proof. In order to understand the proof, it is essential that all numbers be considered to have an implicit "modulo n" following - that is, all numbers are between 0 and $n-1$.

It is easily verified that the above sets are Latin trades, as each column contains only two entries, except for two columns which contain three entries. The entries in the disjoint mate for these columns are forced when the entries for the disjoint mate in columns with two entries are filled in.

Case A. Consider $R_a = L \setminus \{(0, x; x), (2, x-1; x+1)\}$. Then there is no trade $T \subseteq R_a$.

Suppose such a trade T exists with disjoint mate T'.

We prove that none of the entries of R_a can occur in T, starting with column x.

If $(1, x; x+1) \in T$ then $(2, x; x+1) \in T'$. But $(2, x-1; x+1) \notin T$ so $(2, x; x+1) \notin T'$. Therefore $(1, x; x+1) \notin T$. Since each column in a trade must contain at least two entries, $(2, x; x+2) \notin T$.

For $c = x+1$ up to $c = x-2$ in order, we use the following reasoning to show that no elements from column c can be found in T. Since $(2, c-2; c) \notin T$, if $(0, c; c) \in T$ then $(1, c; c) \in T'$. But as $(1, c-1; c) \notin T$, $(1, c; c) \notin T'$ and $(0, c; c) \notin T$. Then, as column c can now contain at most two entries, if $(1, c; c+1) \in T$ then $(2, c; c+1) \in T'$. But $(2, c-1; c+1) \in T$ so $(2, c; c+1) \notin T'$.

Therefore $(1, c; c+1) \notin T$. Since each column in a trade must contain at least two entries (Lemma 3.1 in [9]), $(2, c; c+2) \notin T$.

At the end of this process only two entries remain in column $x-1$ and since any trade must contain at least four entries, such a trade T cannot exist and the proof is complete.

Case B. Consider $R_b = L \setminus \{(0, x; x), (1, x; x+1), (2, x+1; x+3)\}$. Then there is no trade $T \subseteq R_b$. Suppose that such a trade T exists with disjoint mate T'. Then $(2, x; x+2) \notin T$ since column x must contain at least 2 entries. Now if $(0, x+1; x+1) \in T$ then $(1, x+1; x+1) \in T'$. But $(1, x; x) \notin T$ so $(1, x+1; x+1) \notin T'$. Therefore $(0, x+1; x+1) \notin T'$. At this point, for any trade $T \subseteq R_b$, we must have $T \subseteq R_a$ and the proof follows Case A.

Case C. Consider $R_c = L \setminus \{(0, x; x), (1, x+1; x+2), (2, x+1; x+3)\}$. Then there is no trade $T \subset R_c$. Suppose that such a trade T exists with disjoint mate T'. Then $(0, x+1; x+1) \notin T$ since column x must contain at least 2 entries. Now if $(2, x; x+2) \in T$ then $(1, x; x+2) \in T'$. But $(1, x+1; x+2) \notin T$ so $(1, x; x+2) \notin T'$. Therefore $(2, x; x+2) \notin T'$. At this point, for any trade $T \subseteq R_c$, we must have $T \subseteq R_a$ and the proof follows Case A.

Case D. Consider $R_d = L \setminus \{(0, x; x), (1, x-1; x), (2, x-3; x-1)\}$. Then there is no trade $T \subseteq R_d$. Suppose that such a trade T exists with disjoint mate T'. Then if $(2, x-1; x+1) \in T$ then $(2, x-1; x-3) \in T'$. But since $(2, x-3; x-1) \notin T$, then $(2, x-1; x-3) \in T'$. Thus $(2, x-1; x+1) \notin T$. At this point, for any trade $T \subseteq R_d$, we must have $T \subseteq R_a$ and the proof follows Case A.

Case E. Consider $R_e = L \setminus \{(0, x; x), (1, x-2; x-1), (2, x-3; x-1)\}$. Then there is no trade $T \subseteq R_e$. Suppose that such a trade T exists with disjoint mate T'. Since there is only one occurrence of the element $x-1$ in R_e, $(0, x-1; x-1) \notin T$. If $(0, x-2; x-2) \in T$, then $(0, x-2; 0) \in T'$. But since $(0, x; x) \notin T$, $(0, x-2; 0) \notin T'$, and thus $(0, x-2; x-2) \notin T$. Then only $(2, x-2; x)$ is left in column $x-2$ and so $(2, x-2; x) \notin T$. This leaves only one occurrence of the element x in T, and so $(1, x-1; x) \notin T$ and since this leaves only $(2, x-1; x+1)$ in column $x-1$, $(2, x-1; x+1) \notin T$. Then for any trade $T \subseteq R_e$, we must have $T \subseteq R_a$ and the proof follows Case A.

Otherwise, we must show that R contains the trade $T_{even}(x)$ or $T_{odd}(x)$, for some x. Suppose $R = L \setminus \{(0, a; a), (1, b; b+1), (2, c; c+2)\}$.

Take the set of all the trades $T_{even}(x)$ and $T_{odd}(x)$ for the given n and set (a, b, c). We remove the trades which already contain the entries $(0, a; a)$, $(1, b; b+1)$ or $(2, c; c+2)$ and show that there is always at least one trade remaining, except in the cases A, B, C, D and E above.

We consider the case of n odd first. We write that $T_{even}(x, y, z)$ cannot be in R if none of $T_{even}(x), T_{even}(y)$, or $T_{even}(z)$ can be in R.

If $(0, a; a) \notin R$, $T_{even}(a-1, a-2, a-4, \ldots, a-(n-1))$ cannot be in R, and $T_{odd}(a, a-1, a-2, a-3, \ldots, a-(n-2))$ cannot be in R either.

If $(1, b; b+1) \notin R$, $T_{odd}(b-1, b)$ cannot be in R.

If $(2, c, c+2) \notin R$, $T_{even}(c, c-2, c-4, \ldots, c-(n-1))$ cannot be in R, and $T_{odd}(c, c-1, c-3, \ldots, c-(n-2))$ cannot be in R either.

Now we consider which trades are left from the set of trades $T_{even}(x)$. The set of trades we have removed is

$$\begin{aligned} T_{even}(a-1, a-2, a-4, \ldots, a-(n-1)) \cup \\ T_{even}(c, c-2, \ldots, c-(n-1)) \cup T_{even}(b, b-1) \end{aligned}$$

Thus, if we discover that

$$\begin{aligned} &\{a-1, a-2, a-4, \ldots, a-(n-1), c, c-2, \ldots, \\ &c-(n-1), b, b-1\} \\ =&\{0, \ldots, n-1\} \end{aligned}$$

then none of the trades $T_{even}(x)$ are left. (Obviously, some elements on the LHS may occur more than once). But if the equality does not hold, at least one of the trades $T_{even}(x)$ can be found in R.

We let $a = c + y$ and subtract c from every element of the sets on both sides.

$$\begin{aligned} &\{c+y-1, c+y-2, c+y-4, \ldots, c+y-(n-1), c, \ldots \\ &c-2, c-(n-1), b, b-1\} = \{0, \ldots, n-1\} \end{aligned}$$

$$\begin{aligned} \{y-1, y-2, y-4, \ldots, y-(n-1), 0, 1, 3, \ldots, n-4, \\ n-2, b-c, b-c-1\} = \{0, \ldots, n-1\} \end{aligned} \tag{1}$$

Now we let $S = \{y-1, y-2, y-4, \ldots, y-(n-1)\}$ and $T = \{0, 1, 3, \ldots, n-4, n-2\}$. Since $|S \cup T| = |S| + |T| - |S \cap T|$, if $|S \cap T| \geq 4$ then $|S \cup T| \leq |S| + |T| - 4 = n + 1 - 4 = n - 3$. Thus if $|S \cap T| \geq 4$ the size of the LHS of the above equality cannot equal the size of the RHS.

Observe that $|S \cap T| \leq 3$ only for $y = 1, 3, 5, 7, n-3$ or $n-1$.

Next, we consider which trades remain from the set of trades $T_{odd}(x)$. The set of trades we have removed is

$$T_{odd}(a, a-1, a-3, \ldots, a-(n-2)) \cup \\ T_{odd}(c, c-1, c-3, \ldots, c-(n-2)) \cup T_{even}(b, b-1)$$

Thus, if we discover that

$$\{a, a-1, a-3, \ldots, a-(n-2), c, c-1, c-3, \ldots, \\ c-(n-2), b, b-1\} \\ = \{0, \ldots, n-1\}$$

then none of the trades $T_{odd}(x)$ are left. Similarly, if the equality does not hold, at least one of the trades $T_{odd}(x)$ can be found in R.

The above process repeated, setting $a = c + y$, leads to

$$\{y, y-1, y-3, \ldots, y-(n-2), 0, 2, 4, \ldots, \\ n-3, n-1, b-c, b-c-1\} = \{0, 1, \ldots, n-1\} \tag{2}$$

Now we let $S = \{y, y-1, y-3, \ldots, y-(n-2)\}$ and $T = \{0, 2, 4, \ldots, n-3, n-1\}$. Again, if $|S \cap T| \geq 4$ the size of the LHS of the above equality cannot equal the size of the RHS.

To find $|S \cap T|$ here, we need only count the number of even numbers in S. Using similar methods to above, we find that when y is even, $|S \cap T| = \frac{n+1-y}{2}$ and when y is odd, $|S \cap T| = \frac{y+1}{2}$.

Thus, $|S \cap T| \leq 3$ here only for $y = 1, 3, 5, n-5, n-3$ or $n-1$. Taking the results for both $T_{odd}(x)$ and $T_{even}(x)$ into account, we need only consider $y = 1, 3, 5, n-3$ and $n-1$.

We consider $y = 5$ and look at the trades $T_{odd}(x)$. We find the LHS of the equality (2) for the trades $T_{odd}(x)$ gives

$$\{5, 4, 2, 0, \ldots, 9, 7\} \cup \\ \{0, 2, \ldots, n-1, b-c, b-c-1\} = \\ \{0, 2, 4, 5, 6, \ldots, n-1, b-c, b-c-1\}$$

Obviously this can only equal $\{0,\dots,n-1\}$ when $\{b-c,b-c-1\}=\{1,3\}$. Since this isn't possible, there is always a trade in R when $y=5$.

Next, we consider $y=n-3$ and look at the trades $T_{even}(x)$. The LHS of the above equality (1) for the trades $T_{even}(x)$ gives

$$\begin{aligned}\{n-4,n-5,n-7,\dots,2,0,n-2\}\cup\\ \{0,1,\dots,n-2,b-c,b-c-1\}=\\ \{0,1,2,3,\dots,n-4,n-2,b-c,b-c-1\}\end{aligned}$$

Since $\{b-c,b-c-1\}\not\in\{n-3,n-1\}$, there is always a trade in R when $y=n-3$. Thus, there is always a trade in R, except possibly when $y=a-c=1,3$, or $n-1$.

When $y=1$, whether we look at the LHS of equality (1) or (2), all the numbers from 0 to $n-1$ are present. Thus if $y=a-c=1$ (Case A), no trade from the sets $T_{even}(x)\cap T_{odd}(x)$ can be found in R.

When $y=3$, the LHS of equality (1) for $T_{even}(x)$ contains all the numbers from 0 to $n-1$. But the LHS of equality (2) for $T_{odd}(x)$ contains

$$\begin{aligned}\{3,2,0,n-2,n-4,\dots,7,5\}\cup\{0,2,4,\dots,n-2,n-1\}\cup\\ \{b-c,b-c-1\}=\\ \{0,2,3,4,5,6,\dots,n-1,b-c,b-c-1\}.\end{aligned}$$

Thus if $y=3$, only if $b-c=1$ (Case E) or $b-c=2$ (Case D) are there no trades from the sets $T_{even}(x)$ or $T_{odd}(x)$ in R.

When $y=n-1$, the LHS of equality (2) for $T_{odd}(x)$ contains all the numbers from 0 to $n-1$. But the LHS of equality (1) for $T_{even}(x)$ contains

$$\begin{aligned}\{n-2,n-3,n-5,\dots,2,0\}\cup\{0,1,3,\dots,n-4,n-2\}\cup\\ \{b-c,b-c-1\}=\\ \{0,1,2,3,\dots,n-4,n-3,n-2\}.\end{aligned}$$

Thus if $y=n-1$, then only if $b-c=-1$ (Case B) or $b-c=0$ (Case B) are there no trades from $T_{even}(x)$ or $T_{odd}(x)$ in R.

This completes the proof for n odd.

We now consider n even. The notation is the same as in the proof for n odd, but the proof is quite different, and requires division into many cases.

There are four cases to consider, depending on the parities of a and c.

Case 1. a and c are even.

Considering only the trades $T_{odd}(x)$, we remove the trades $T_{odd}(a-1, a-2, a-3, b, b-1, b-2, b-3, c, c-1, c-2)$. Now if all the trades have been removed, we will have

$$\begin{aligned}&\{a-1, a-2, a-3, b, b-1, b-2, b-3, c, c-1, c-2\}\\ =&\{0, \ldots, n-1\}\end{aligned}$$

But since $n \geq 12$ and there are at most 10 different numbers on the LHS, this cannot be an equality: there must be some trades from $T_{odd}(x)$ remaining in R.

Case 2. a and c are odd.

Considering only the trades $T_{even}(x)$, we remove the trades $T_{even}(a-1, a-2, a-3, b, b-1, b-2, b-3, c, c-1, c-2)$. Similarly to Case 1, we will have an equation with at most 10 different numbers on the LHS, and so there must still be some trades from $T_{even}(x)$ in R.

Case 3. a is even and c is odd.

Considering the trades $T_{even}(x)$, we remove the trades $T_{even}(a, a-1, a-3, a-4, \ldots, a-(n-1), b, b-1, b-2, b-3, c, c-1, c-2)$. Using the familiar equality, this set of numbers can only not be equal to $\{0, \ldots, n-1\}$ if $a \notin \{b+2, b+1, b, b-1, c+2, c+1, c\}$. We know from the parities of a and c that $a \notin \{c, c+2\}$. Thus there is a trade in R if $a \notin \{b+2, b+1, b, b-1, c+1\}$.

Considering the trades $T_{odd}(x)$, we remove the trades $T_{odd}(a-1, a-2, a-3, b, b-1, b-2, b-3, c, c-2, c-3, \ldots, c-(n-1))$. This set of numbers can only be not equal to $\{0, \ldots, n-1\}$ if $c-1 \notin \{a-1, a-2, a-3, b, b-1, b-2, b-3\}$, that is, if $c \notin \{a, a-1, a-2, b+1, b, b-1, b-2\}$. We know from the parities of a and c that $c \notin \{a, a-2\}$. Thus there is a trade in R if $c \notin \{a-1, b+1, b, b-1, b-2\}$.

Taking both of these results together, we know that if $a = c+1$ (Case A) there is no trade. Also, there is no trade if $b \notin \{a-2, a-1, a, a+1\}$ or $b \notin \{c-1, c, c+1, c+2\}$. Combining these results, we have sixteen cases where there are no trades, all covered in Cases A to E above, as follows.

- $b = c-1 = a-2$. $a = c+1$, Case A.
- $b = c-1 = a$. No trade, Case B.
- $b = c = a-1$. $a = c+1$, Case A.
- $b = c = a+1$. No trade, Case C.

- $b = c + 1 = a - 2$. No trade, Case E.
- $b = c + 1 = a$. $a = c + 1$, Case A.
- $b = c + 2 = a - 1$. No trade, Case D.
- $b = c + 2 = a + 1$. $a = c + 1$, Case A.

The remaining cases are discounted by parity contradictions, with variables from $\{a, b, c\}$ required to be both simultaneously odd and even.

Case 4. a is odd and c is even.

Considering the trades $T_{even}(x)$, we remove the trades $T_{even}(a-1, a-2, a-3, b, b-1, b-2, b-3, c, c-2, c-3, \ldots, c-(n-1))$. Thus we have a trade in R if $c - 1 \notin \{a-1, a-2, a-3, b, b-1, b-2, b-3\}$, that is, if $c \notin \{a, a-1, a-2, b+1, b, b-1, b-2\}$. But $c \notin \{a, a-2\}$. Thus if $c \notin \{a-1, b-1, b, b+1, b-2\}$ there is a trade in R.

Considering the trades $T_{odd}(x)$, we remove the trades $T_{odd}(a, a-1, a-3, \ldots, a-(n-1), b, b-1, b-2, b-3, c, c-1, c-2)$. Thus we have a trade in R if $a-2 \notin \{b, b-1, b-2, b-3, c, c-1, c-2\}$, that is, if $a \notin \{b+2, b+1, b, b-1, c+2, c+1, c\}$. But $a \notin \{c, c+2\}$. Thus if $a \notin \{b+2, b+1, b, b-1, c+1\}$ there is a trade in R.

As this is the same result as for Case 3, the chopped rectangles in Cases A to E are the only ones containing no trades.

This completes the proof for n even.

Corollary 1 *Of the n^3 possible $3 \times n$ chopped Latin rectangles of order n, exactly $n(n-4)$ contain no trade.*

4 Conclusion

How might Conjecture 2 be proved? Essentially, there are only two ways: a proof of the existence of a trade in the remaining entries or an algorithm to generate such a trade. An algorithm may prove to be more difficult than the complex approach of Cavenagh [5], and so as more is known about trades on three rows, it may prove simpler to try the existence approach. Perhaps such a proof will show that if no trades can exist in any pair or triple of rows from the four rows, a trade must exist using all four rows. As the value of $scs(n)$ is now known for $n \leq 8$, it may simplify matters to concentrate only on values of $n \geq 9$, where there are more trades contained in each chopped rectangle.

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