A new bound on the size of the largest critical set in a Latin square

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Abstract

A critical set in an $n \times n$ array is a set C of given entries, such that there exists a unique extension of C to an $n \times n$ Latin square and no proper subset of C has this property. The cardinality of the largest critical set in any Latin square of order n is denoted by $\mathrm{lcs}(n)$. In 1978 Curran and van Rees proved that $\mathrm{lcs}(n) \leq n^2 - n$. Here we show that $\mathrm{lcs}(n) \leq n^2 - 3n + 3$.

Keywords: Latin squares, largest critical sets, intercalates

1 Introduction

For the purposes of this paper, a Latin square of order n is an $n \times n$ array of integers chosen from the set $X = \{1, 2, \ldots, n\}$ such that each integer occurs exactly once in each row and exactly once in each column. An example of a Latin square of order 4 is shown below.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

A Latin square can also be written as a set of ordered triples $\{(i, j; k) \mid \text{symbol } k$ occurs in position (i, j) of the array$\}$.

A partial Latin square P of order n is an $n \times n$ array with entries chosen from the set $X = \{1, 2, \ldots, n\}$, such that each element of X occurs at most once in each row and at most once in each column. Hence there are cells in the array that may be empty, but the positions that are filled have been filled so as to conform with the Latin property of the array. Let P be a partial Latin square of order n. Then $|P|$ is said to be the size of the partial Latin square and the set of positions $\mathcal{S}_P = \{(i, j) \mid (i, j; k) \in P\}$ is said to determine the shape of P.

A partial Latin square C contained in a Latin square L is said to be uniquely completable if L is the only Latin square of order n with k in position (i, j) for every $(i, j; k) \in C$. A critical set C contained in a Latin square L is a partial Latin square that is uniquely completable and no proper subset of C satisfies this requirement. The name "critical set" and the concept were invented by a statistician, John Nelder, about 1977, and his ideas were first published in a note [**15**]. This note posed the problem of giving a formula for the size of the largest and smallest critical sets for a Latin square of a given order. Curran and van Rees [**6**], and independently Smetaniuk [**17**] were the first papers written on the subject. See [**12**] and [**2**] for further details. Let $\text{lcs}(n)$ denote the size of the largest critical set and $\text{scs}(n)$ the size of the smallest critical set in any Latin square of order n. It was conjectured by Nelder [**16**] that $\text{lcs}(n) = (n^2 - n)/2$, and by Nelder [**16**] and also by one of the present authors [**14**] and Bate and van Rees [**2**], independently, that $\text{scs}(n) = \lfloor n^2/4 \rfloor$. The equality for $\text{lcs}(n)$ was shown to be false in 1978, when Curran and van Rees, [**6**], found that $\text{lcs}(4) \geq 7$. Unfortunately, the research over the last twenty years has not added much information and in general an upper bound is given by $n^2 - n$. In this paper we show that $\text{lcs}(n) \leq n^2 - 3n + 3$.

In order to validate the construction we require the definition of a Latin interchange and an associated lemma.

Let P and P' be two partial Latin squares of the same order and with the same shape. Then P are P' are said to be mutually balanced if the set of entries in each row (and column) of P are the same as those in the corresponding row (and column) of P'. They are said to be disjoint if no position in P' contains the same entry as the corresponding position in P. A Latin interchange I is a partial Latin square for which there exists another partial Latin square I', of the same order, the same shape and with the property that I and I' are disjoint and mutually balanced. The partial Latin square I' is said to be a disjoint mate of I (see [**9**] and [**12**] for more references). An example of a Latin interchange and its disjoint mate is given below.

	2	3
1		2
2	3	1

I

	3	2
2		1
1	2	3

I'

The following lemma clarifies the connection between critical sets and Latin interchanges.

Lemma 1.1 *A partial Latin square $C \subseteq L$, of size s and order n, is a critical set for a Latin square L if and only if the following hold:*

(1) *C contains at least one element of every Latin interchange that is contained in L;*

(2) *for each $(i,\ j;\ k) \in C$, there exists a Latin interchange I_r contained in L such that $I_r \cap C = \{(i,\ j;\ k)\}$.*

Proof. (1) If C does not contain an element from some Latin interchange I in L, where I has a disjoint mate I', then C is also a partial Latin square of $L' = (L \setminus I) \cup I'$. Hence C is not uniquely completable.

(2) Since C is a critical set, $C \setminus \{(i,\ j;\ k)\}$ is not uniquely completable. Therefore $C \setminus \{(i,\ j;\ k)\}$ may be completed in at least two different ways, thus there exists a Latin interchange $I_r \subseteq L$ such that $I_r \cap C = \{(i,\ j;\ k)\}$. ■

For a critical set C in a Latin square L we define sets for each row i, column j and element k. Let $R_i = \{k|\ (i,j;k) \in C\}$, $C_j = \{k|\ (i,j;k) \in C\}$, and $E_k = \{(i,j)|\ (i,j;k) \in C\}$. So R_i (C_j) is the set of elements which appear in row i (column j) and E_k is the set of positions where the element k appears.

2 The value of $\text{lcs}(n)$ for small n

In the following table some known values of $\text{lcs}(n)$ are listed for small values of n. The extra columns are to compare different bounds discussed in this paper.

n	$\text{lcs}(n)$	n^2-3n+3	$\lfloor n^2-n^{3/2}\rfloor$	$\lfloor(1-(\frac{3}{4})^{log_2 n})n^2\rfloor$
1	0	1	0	0
2	1	1	1	1
3	3	3	3	3
4	7	7	8	7
5	11	13	13	12
6	18	21	21	18
7	≥ 25	31	30	27
8	≥ 37	43	41	37
9	≥ 44	57	54	48
10	≥ 57	73	68	61

The values listed for $\text{lcs}(n)$, expect for $n = 5, 7$, 9, and 10, are given in [**7**]. The value for $n = 5$ and the bound for $n = 7$ were given by A. Khodkar [**13**]. In the Appendix we give some examples for the largest known critical sets for $n = 5, 7$, 9, and 10. The value for $n = 6$ is given in [**1**].

3 Non-critical sets

The following lemma is our main tool in improving the upper bound on the possible size of $\text{lcs}(n)$.

Lemma 3.1 *Let C be a critical set for a Latin square L and assume that there exists i such that $|R_i| = n-1$. Then the missing element in row i does not occur anywhere in C, and the column corresponding to the missing element is empty. That is, if $(i, j; k) \in L \setminus C$, then $|C_j| = |E_k| = 0$.*

Proof. Without loss of generality, let $i = 1$ and assume that C contains the elements $\{(1, x; x)|\ 1 \leq x \leq n-1\}$ and that position $(1, n)$ is empty. Note that the element n may not appear in column n in C, else no element could be placed in position $(1, n)$ of L.

By Lemma 1.1 part (2), for each x $(1 \leq x \leq n-1)$ there exists a Latin interchange $I_x \subseteq L$ such that $I_x \cap C = \{(1, x; x)\}$. Since there is only one empty position in the first row, it follows that $\{(1, x; x), (1, n; n)\} \subseteq I_x$. Now the interchange I_x has a disjoint mate, say I'_x. In this case since $(1, x; n) \in I'_x$, for some r, $(r, x; n) \in I_x$, and since $|I_x \cap C| = 1$, $(r, x; n) \in L \setminus C$. So n does not occur in column x in C. Since x ranges over all columns from 1 to $n-1$, n does not occur in C at all. Therefore $|E_n| = 0$.

Also we have $(1, n; x) \in I'_x$. Thus for some s, $(s, n; x) \in I_x$. Similarly we have $(s, n; x) \notin C$; therefore no element apart from n may occur in column n in C, and we have said that n does not occur in column n either. Therefore column n is empty. So $|C_n| = 0$. ■

We can generalize Lemma 3.1 to the following.

Lemma 3.2 *Let C be a critical set for a Latin square L and assume that there exists i, such that $|R_i| = n - m$, where $\{(i, c_1; e_1), (i, c_2; e_2), \ldots, (i, c_m; e_m)\} \subseteq L \setminus C$ and $\{(i, c_{m+1}; e_{m+1}), \ldots, (i, c_n; e_n)\} \subseteq C$. Then we have*

(1) *In each of the columns $c_{m+1}, c_{m+2}, \ldots, c_n$ in C, at least one of the elements $e_1, e_2, \ldots, e_m$ is missing. That is for each $x \in \{c_{m+1}, c_{m+2}, \ldots, c_n\}$, there exists an element $y \in \{e_1, e_2, \ldots, e_m\}$, and a row $r \in \{1, 2, 3, \ldots, n\} \setminus \{i\}$ such that $(r, x; y) \in L \setminus C$.*

(2) *For each element $e \in \{e_{m+1}, e_{m+2}, ..., e_n\}$, we have a column $c \in \{c_1, c_2, \ldots, c_m\}$, from which this element is missing.*

Proof. (1) Without loss of generality we may assume that $i = 1$ and $c_j = e_j = j$ for $j = 1, 2, \ldots, n$. For each $x \in \{m+1, m+2, \ldots, n\}$, there exists a Latin interchange I_x such that $I_x \subseteq L$ and $I_x \cap C = \{(1, x; x)\}$. So if I'_x is the disjoint mate of I_x then there exists $y \in \{1, 2, \ldots, m\}$ such that $(1, x; y) \in I'_x$, implying that there exists $r \in \{2, \ldots, n\}$ such that $(r, x; y) \in I_x$. Since $|I_x \cap C| = 1$, $(r, x; y) \in L \setminus C$.

(2) Similarly for each $e \in \{m+1, m+2, \ldots, n\}$, there exists a Latin interchange I_e such that $I_e \subseteq L$ and $I_e \cap C = \{(1, e; e)\}$. So if I'_e is the disjoint mate of I_e then there exists $c \in \{1, 2, \ldots, m\}$ such that $(1, c; e) \in I'_e$, implying that there exists $s \in \{2, \ldots, n\}$ such that $(s, c; e) \in I_e$. Since $|I_e \cap C| = 1$, $(s, c; e) \in L \setminus C$. ■

Theorem 3.1 *If C is a uniquely completable partial Latin square of order n completing to the Latin square L with $|C| > n^2 - 3n + 3$, then C is not a critical set.*

Proof. We prove this result by contradiction. Suppose C is a critical set. Since a critical set in a Latin square of order n cannot have n triples whose i-th components are the same $(1 \leq i \leq 3)$ (see for example [**6**]), we can assume that any row or column contains at most $n-1$ elements and any element occurs at most $n-1$ times.

We have three cases to consider.

Case 1 There exists a row i such that $|R_i| = n-1$. Assume that $(i,j;k) \in L \setminus C$. Then by Lemma 3.1, $|C_j| = |E_k| = 0$. Now if there exists j' ($j' \neq j$) such that $|C_{j'}| = n-1$ and $(i',j';k') \in L \setminus C$, then we have $|R_{i'}| = 0$. These together imply that $|C| \leq n^2 - (2n-1) - (n-2) = n^2 - 3n + 3$. Otherwise $|C_l| \leq n-2$, for all $l \neq j$, and $|C_j| = 0$; and thus $|C| \leq (n-1)(n-2) = n^2 - 3n + 2$.

Case 2 For all i ($1 \leq i \leq n$) we have, $|R_i| \leq n-3$. Then $|C| \leq n(n-3) = n^2 - 3n$.

Case 3 For all i ($1 \leq i \leq n$) we have $|R_i| \leq n-2$ and there exists a row r such that $|R_r| = n-2$. And by symmetry we may assume that for all j ($1 \leq j \leq n$) we have $|C_j| \leq n-2$. Assume that $R_r = \{e_3, e_4, \ldots, e_n\}$, and $\{(r, c_1; e_1), (r, c_2; e_2)\} \subset L \setminus C$. Then by Lemma 3.2 each of the elements $e_3, e_4, \ldots, e_n$ occurs at most once in columns c_1 and c_2. This means $|C_{c_1}| + |C_{c_2}| \leq n$. Thus $|C| \leq n(n-2) - (n-4) = n^2 - 3n + 4$. We will show that $|C| = n^2 - 3n + 4$ is also impossible. Proof of this fact is somewhat involved and we need to introduce more notation.

First note that if we consider the conjugate of the Latin square L we may assume that for all k ($1 \leq k \leq n$) we have $|E_k| \leq n-2$. Let $f_k = n - 2 - |E_k|$. We have $f_k \geq 0$, for all k ($1 \leq k \leq n$). Assume $|C| = n^2 - 3n + 4$. Then

$$\sum_{k=1}^{n} f_k = n(n-2) - |C| = n - 4.$$

For each position (i,j), $1 \leq i,j \leq n$, we define $x_{i,j} = |R_i \cup C_j|$. We have

$$(*) \qquad \sum_{1 \leq i,j \leq n} x_{i,j} = n^3 - \sum_{k=1}^{n} (n - |E_k|)^2.$$

In fact for each position (i,j), $1 \leq i,j \leq n$, we have $x_{i,j} = n$, except when an element k is missing from *both* row i and column j in C. For each k we have exactly $(n - |E_k|)^2$ such positions. They are the positions which are in the $(n - |E_k|) \times (n - |E_k|)$ subsquare obtained from the $n \times n$ array by omitting all the rows and columns containing element k in C. Each such position causes a "-1" in the summation of the left hand side of $(*)$.

Note that since C is a critical set, for each position $(i,j) \in L \setminus C$, that is for each position in L in which C is empty, we have $x_{i,j} \leq n-1$. Thus

$$\begin{aligned}
\tfrac{1}{|C|} \sum_{(i,j) \in C} x_{i,j} &= \tfrac{1}{|C|} \Big((n^3 - \sum_{k=1}^{n} (n - |E_k|)^2) - \sum_{(i,j) \in L \setminus C} x_{i,j} \Big) \\
&\geq \tfrac{1}{n^2 - 3n + 4} \Big((n^3 - \sum_{k=1}^{n} (f_k + 2)^2) - (3n-4)(n-1) \Big) \\
&= \tfrac{1}{n^2 - 3n + 4} (n^3 - 3n^2 - n + 12 - \sum_{k=1}^{n} f_k^2).
\end{aligned}$$

where by $(i, j) \in C$ we mean a position in C which is not empty.
Since $\sum_{k=1}^{n} f_k^2 \leq (\sum_{k=1}^{n} f_k)^2 = (n-4)^2$, thus $\frac{1}{|C|} \sum_{(i,j)\in C} x_{i,j} \geq \frac{n^3-3n^2-n+12-(n-4)^2}{n^2-3n+4} = n-1$.
This implies that, either

(i) for some position $(i, j) \in C$ we have $x_{i,j} > n-1$; or

(ii) for all $(i, j) \in C$, $x_{i,j} = n-1$.

The first case is contradictory with C being a critical set. In the second case if we remove an element $(a, b; e) \in C$, then we have

- $x_{a,b} = n-2$ and $x_{a,j}, x_{i,b} \leq n-1$, for all (a, j) and $(i, b) \in C$; and
- $x_{i,j} = n-1$; for any other pair $(i, j) \in C$.

But if case (ii) holds, then all of the inequalities that we have above must be equalities, and this implies that for every $(i, j) \in L \setminus C$, we have $x_{i,j} = n-1$. This follows because we have used the inequality $x_{i,j} \leq n-1$. So $C \setminus \{(a, b; e)\}$ can be completed to L, first by completing any position not in the row a or column b, then the positions of row a and column b. This is a contradiction. ∎

4 Conjectures and Questions

There are some conjectures and questions which arise from this research and we discuss them in this section.

Conjecture 1 $\mathrm{lcs}(n) \leq n^2 - n^{3/2}$.

This is motivated by the proof of Theorem 3.1. It is analogous to a similar conjecture made by Brankovic, Horak, Miller, and Rosa, in [**5**], concerning the size of the largest premature partial Latin square.

Conjecture 2 $\mathrm{lcs}(n) \leq (1 - (\frac{3}{4})^{log_2 n})n^2$.

This is true for the current known values of $\mathrm{lcs}(n)$. It implies that $\mathrm{lcs}(2^n) = 4^n - 3^n$. This conjecture is based on Stinson and van Rees's result in [**18**] that $\mathrm{lcs}(2^n) \geq 4^n - 3^n$. We postulate that this is an equality.

Question 1 *If C is a critical set of order n and of size $\mathrm{lcs}(n)$, do there exist i, j, k, $1 \leq i, j, k \leq n$, such that $|R_i| = |C_j| = |E_k| = 0$? That is, is there always an empty row, an empty column, and a missing element in a critical set of size $\mathrm{lcs}(n)$?*

Evidence for the "yes" case in Question 1 is that every critical set of largest size in Latin squares of orders 1 to 6 has this property. Every example in Stinson and van Rees [**18**] and in Donovan [**7**] where critical sets of largest known size are given, has this property. All the constructions given for large critical sets given in such articles as [**8**],[**10**],[**16**] and [**18**] have this property. However, the example of a critical set of largest known size in a Latin square of order 10, given in Appendix 1, does not have this property.

A Latin interchange of size 4 is said to be an intercalate, and the largest number of intercalates in any Latin square of order n is denoted by $I(n)$ (see [**11**]). Below, we ask how $I(n)$, the maximum number of intercalates in an $n \times n$ Latin square, and $\mathrm{lcs}(n)$ are related.

Question 2 *If C is a critical set for the Latin square L of order n and size* $\mathrm{lcs}(n)$*, does L have $I(n)$ intercalates?*

Question 3 *If L is a Latin square of order n with $I(n)$ intercalates, does L contain a critical set C of size* $\mathrm{lcs}(n)$*?*

Appendix

Here we give some examples for the largest known critical sets for $n = 5, 7$, 9, and 10.

A critical set of order 5 and size 11:

2		4	3	
		1	2	
	2	3	1	
3	1	2		

A critical set of order 7 and size 25:

	3	2	1		5	
6		3	5	4	1	
	6	5	4	3	2	
		4	3	5		
3	4	1	2		6	
1		6			3	

In the critical set of order 5, an instance where $\forall i, |R_i| \leq n-2$ has been given to show that where C is a critical set of size lcs(n), it is not necessary to have some i, j, k; $1 \leq i, j, k \leq n$, such that $|R_i| = n-1$ and $|C_j| = n-1$ and $|E_k| = n-1$.

Above, we also gave a similar example for the critical set of order 7, though it is not known whether lcs$(7) = 25$. And a critical set of order 9 and size 44 is given below which also has the same property:

1		3		5			7	
	1	2				6	5	
3	2	1			6	5	8	
			1		2	3	4	
5			2	1	4	7	3	
		5	3	2	1	4	6	
	6	7	4	3		1	2	
7	5	6	8	4	3	2	1	

Critical sets of order 9 for all sizes from 20 to 44 inclusive are known to exist (see [**7**] and [**3**]).

A critical set of order 10 and size 57:

1		3		5		7		9	
	1	2			5		6	8	
3	2	1			9	6	7	5	
			1	2	3		8	4	
5			2	1	10	4	3		
	5	9	3	10	1	2		6	
7		6		4	2	1	5	3	
	6	7	8	3		5	1	2	
9	8	5	4		6	3	2	1	

Critical sets of order 10 for all sizes from 25 to 57 inclusive are known to exist (see [**7**], [**4**], and [**3**]).

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