

# The size of the smallest uniquely completable set in order 8 Latin squares

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## Abstract

In 1990, Kolesova, Lam and Thiel determined the 283,657 main classes of Latin squares of order 8. Using techniques to determine relevant Latin trades and integer programming, we examine representatives of each of these main classes and determine that none can contain a uniquely completable set of size less than 16. In three of these main classes, the use of trades which contain less than or equal to three rows, columns, or entries does not suffice to determine this fact. We closely examine properties of representatives of these three main classes. Writing the main result in Nelder's notation for critical sets, we prove that  $\text{scs}(8)=16$ .

## 1 Introduction

A *Latin square*  $L$  of order  $n$  is an  $n \times n$  array of entries  $\{(i, j; k)\}$  such that each row and column of  $L$  contains each of  $n$  possible elements exactly once. We will refer to the Latin square based on the group table of  $\mathbb{Z}_n$  simply as  $\mathbb{Z}_n$ . A *uniquely completable (UC)* set  $U$  is a subset of a Latin square  $L$  such that  $L$  is the only superset of  $U$  which is a Latin square. A *critical set*  $C$  of  $L$  is a subset of  $L$  such that  $C$  is uniquely completable and no subset of  $C$  has this property. Critical sets were introduced by Nelder [16].

A related concept is that of the *Latin trade*. In this paper, we will consider a Latin trade to be the set of entries in which two Latin squares of the same order differ. An *intercalate* is a Latin trade of size 4. The connection between uniquely completable sets and Latin trades is well-known, for example see [12], [8]. It is expressed in the following lemma.

**Lemma 1** *In any uniquely completable set  $U$  for a Latin square  $L$ , each trade in  $L$  must intersect  $U$  in at least one position.*

$\text{scs}(n)$  is a function defined by Nelder as the smallest size of a critical set in any  $n \times n$  Latin square. Nelder determined that  $\text{scs}(3)=2$ , Curran and van Rees [6] that  $\text{scs}(4)=4$  and  $\text{scs}(5)=6$ , Bate and van Rees [2] that  $\text{scs}(6)=9$ , and Adams and Khodkar [1] that  $\text{scs}(7)=12$ . Here we show that  $\text{scs}(8)=16$ .

For a set  $S$  in a Latin square  $L$  we define sets for each row  $i$ , column  $j$  and element  $k$ . Let  $R_i(S) = \{k \mid (i, j; k) \in S\}$ ,  $C_j(S) = \{k \mid (i, j; k) \in S\}$ , and  $E_k(S) = \{(i, j) \mid (i, j; k) \in S\}$ . So  $R_i(S)$  ( $C_j(S)$ ) is the set of elements which appear in row  $i$  (column  $j$ ) of  $S$  and  $E_k(S)$  is the set of positions where the element  $k$  appears in  $S$ .

Now define  $R(S) = |\{i \mid R_i(S) \neq \emptyset\}|$ ,  $C(S) = |\{j \mid C_j(S) \neq \emptyset\}|$ ,  $E(S) = |\{k \mid E_k(S) \neq \emptyset\}|$ .

We will write that a trade  $T$  has  $R(T)$  rows,  $C(T)$  columns and  $E(T)$  elements.

Based on results by Cavenagh [5], and considering the paper by Horak, Fleischner, and Aldred [10], the author decided to conjecture the following.

**Conjecture 1** *In a subset  $S$  of  $L$ , a Latin square of order  $n \geq 3$ ,  $S \leq \lfloor \frac{n^2}{4} \rfloor$ , there exists a Latin trade  $T \in L$  such that  $T \cap S = \emptyset$  and  $R(T) \leq 3, C(T) \leq 3$ , or  $E(T) \leq 3$ .*

## 2 Integer programming methods

A survey of defining sets by Donovan et al [8] includes a description of an algorithm called ‘‘Algorithm B’’ and the *minimising* technique reproduced verbatim here. Smallest defining sets for designs are a concept analogous to smallest critical sets for Latin squares, and blocks are here analogous to entries in Latin squares.

**Algorithm 1 (B)** *Find some trades in  $D$  and put these in a list  $\mathcal{T}$ . Now perform steps (1) and (2) repeatedly until  $S$  has only one completion.*

(1) *Form the integer programme corresponding to  $\mathcal{T}$ , find an optimal solution, and form the set of blocks  $S$  corresponding to this solution.*

(2) *If  $S$  does not complete uniquely, then completions not equal to  $D$  define additional trades in  $D$ , and these are added to  $\mathcal{T}$ .*

*The running time for step (1) can be improved by minimising  $\mathcal{T}$ . That is, if  $T_a \in \mathcal{T}$  and  $T_b \in \mathcal{T}$ , and if  $T_a \subsetneq T_b$ , then  $T_b$  can be removed from  $\mathcal{T}$ .*

This algorithm is suitable for finding a smallest defining set for a design, but it is not suitable for finding the size of the smallest critical set in Latin squares of order 8, as it requires too much computation time. In this paper, we are concerned only with the *size* of the smallest critical set.

In the paper by Donovan et al it is stated that ‘‘Algorithm B yields only a single smallest defining set’’. However, it can be easily extended with a third step, as follows, to find all smallest defining sets.

(3) If  $S$  has unique completion and contains  $s$  blocks, add an additional constraint  $C$  to the integer programme which ensures that less than  $s$  of the blocks in  $S$  can occur. Go back to step (1).

Similarly to Algorithm B, if  $\mathcal{T}$  is the complete set of Latin trades in a Latin square  $L$ , then the optimal solution to the following integer programme is the size of the minimal critical set in  $L$ .

Minimize:  $\sum_{x \in L} C_x$   
 Subject to:  
 for each  $T \in \mathcal{T}$ ,  $\sum_{x \in T} C_x \geq 1$   
 where  $C_x$  is 1 if  $x \in C$  and 0 otherwise.

The application of Conjecture 1 means that in order to determine the value of  $\text{scs}(n)$  for small values of  $n$ , we should first find representatives of each main class of Latin square of order  $n$  and for each square, find the Latin trades on 3 rows, columns and elements and write a 0-1 integer programme as above. However, for our purposes it is not necessary to calculate the exact value of the smallest critical set for every main class. In general, it was found that limiting the size of the trades to a value determined by trial and error and considering trades only with less than or equal to 3 rows, columns, or elements was effective. Thus the following algorithm was used for a Latin square  $L$  of order  $n$ .

**Algorithm 2** *Find all trades with less than or equal to 3 rows, columns, or elements. This is achieved by considering all completions of  $L \setminus S$ , where  $S$  is a set of 3 rows, columns, or elements of  $L$ . When  $S$  ranges over all possible sets of 3 rows, columns, or elements of  $L$ , then we can find all such trades by considering the difference between  $L$  and each possible completion. See Bean [4] for completion methods. Minimize the list of trades as in Algorithm B.*

*In the following, the initial value of  $z$  and the increment are chosen to minimize running time.*

(1) *Create an integer programme IP as above using the trades of size less than  $z$ . (Adding an extra constraint to ensure that the sum of the  $C'_x$ s does not equal or exceed  $\lfloor \frac{n^2}{4} \rfloor$  speeds up the process.) Solve IP.*

(2) *If IP has a solution with size  $< \lfloor \frac{n^2}{4} \rfloor$  then increase  $z$  and repeat step 1, until  $z = 3n$ . Otherwise stop.*

(3) *Similarly to the new step (3) above, check the solution  $S$  of size  $s < \lfloor \frac{n^2}{4} \rfloor$  for unique completion. If it does not have unique completion, remove it from future consideration by adding a constraint which ensures that less than  $s$  of the cells of  $S$  are considered in future iterations.*

We used CPLEX [11] or BonsaiG [9] to solve the integer programme.

$\text{scs}(n)$  may be determined by considering all the main classes of Latin squares of order  $n$ , as Donovan et al [7] showed. Kolesova, Lam and Thiel [13] determined the 283,657 main classes of Latin squares of order 8. These, and the main classes of smaller orders, were obtained from McKay's website [14].

For  $n = 6$ , using Algorithm 2 and limiting the size of the trades to 10 or less, we can determine that  $\text{scs}(6)=9$  in approximately 5 seconds on an Athlon 1200Mhz computer. Twelve main classes must be examined, thus each main class takes about half a second. For  $n = 7$ , by limiting the size of the trades to 11 or less, we can determine that  $\text{scs}(7)=12$  in 34.5 minutes on the same computer. We examine 147 main classes, thus each main class takes about 12 seconds.

As might be expected, there is a strong correlation between the number of intercalates in a Latin square of order  $n$  and the time taken to prove the non-existence in the square of a critical set of size less than  $\lfloor \frac{n^2}{4} \rfloor$ . For example, there are two main classes of  $7 \times 7$  Latin squares with no intercalates; in the computation mentioned above, one of these main classes,  $\mathbb{Z}_7$ , accounts for 13.8 minutes of the total 34.5 minute computation. The other main class contains 12 Latin trades of order 6 (the size of the smallest trade in  $\mathbb{Z}_7$  is 9) and accounts for 2.2 minutes. The one main class with 1 intercalate takes 1.1 minutes, and every other main class takes 45 seconds or less. In the  $6 \times 6$  Latin squares, the effect is less apparent, but the four main classes with less than  $\lfloor \frac{6^2}{4} \rfloor$  intercalates are the four Latin squares which take the most time.

For a given order 8 Latin square, it is much easier to test whether a critical set of size less than 16 exists than it is to find the size of the smallest critical set in that square. This is strictly a constraint satisfaction problem rather than an integer programming problem, because the minimization step is not necessary. Increasing the number of constraints, for this problem, seems to in general slow solution down. We began by limiting the size of the trades to 10, which was sufficient for more than 95% of Latin squares of order 8, before increasing the limit to 12, 14 and then 24.

For the first 138,800 main classes (arranged in descending order of the number of intercalates) the computers used were an Athlon 1200Mhz, and computers with 5 Athlon MP 1667Mhz CPUs at IPM, Tehran. CPLEX was used. After this a network at the University of Queensland consisting of 128 Sun Fireblades and 33 dual Pentium III 800 MHz computers were used with BonsaiG to complete the calculation, with completion of some gaps by a Pentium IV 2.4 Ghz at IPM. The programmes written can be obtained by emailing the author. The first part of computation took about three months, and the second part about one month.

As Conjecture 1 was true for  $n = 7$  it was conjectured that it would be true for all  $n$ . However, while determining the main result of this paper, the author discovered the following three main classes  $X$ ,  $Y$ , and  $W$  (see Table 1). In  $X$  there exist 4 sets  $X_1, X_2, X_3$ , and  $X_4$ , each of size 15, such that for each Latin trade  $T$  in  $X_i, 1 \leq i \leq 4$  such that  $T \cap X_i = \emptyset$ ,  $R(T), C(T)$  and  $E(T)$  are all greater than 3. In other words, for each of these sets, every trade in the square with less than or equal to three rows, columns, or elements intersects this set. Similarly, in  $Y$  there exist 12 sets of size 15  $Y_1, \dots, Y_{12}$  with this property, and in  $W$  the set of size 15,  $W_1$ . These are the only three  $8 \times 8$  main classes with this property, thus step (3) of Algorithm 2 is used only for these main classes.

These sets are presented in Tables 2, 3 and 4.

Thus even if a method could be found to generalize the trades constructed by Cavenagh to all trades on three rows, columns, or elements, such a method would not be enough to prove that  $\text{scs}(n) = \lfloor \frac{n^2}{4} \rfloor$ .

### 3 Properties of the exceptional main classes

1	6	3	4	5	2	8	7
5	7	1	2	3	8	6	4
4	5	2	8	6	7	3	1
3	2	8	7	1	6	4	5
2	8	4	6	7	1	5	3
7	4	6	3	8	5	1	2
8	1	7	5	4	3	2	6
6	3	5	1	2	4	7	8

1	6	3	4	5	2	8	7
3	5	2	7	4	8	6	1
4	3	7	8	6	2	1	5
2	1	8	6	7	3	5	4
5	7	1	2	8	6	4	3
7	8	6	1	5	4	3	2
8	4	5	3	1	7	2	6
6	2	4	5	3	1	7	8

8	6	7	4	1	2	3	5
7	3	2	8	5	4	1	6
6	4	5	3	7	1	2	8
2	5	3	1	8	6	4	7
5	8	1	2	6	3	7	4
1	7	4	6	3	8	5	2
3	2	8	5	4	7	6	1
4	1	6	7	2	5	8	3

X

Y

W

Table 1: The three main classes  $X$ ,  $Y$ , and  $W$

	6	3					
5		1					
				7	3		
			7	1			
2			6				
	4			5			
			4		2		
							8

		3		5			
5						6	
4					7		
	2			1			
			6		1		
	4		3				
		7				2	
							8

		3			2		
5	7						
				6	7		
3				1			
		4	6				
	4					1	
			5			2	
							8

1			4				
			2	3			
	5	2					
					6	4	
				7		5	
7		6					
	1				3		
							8

Table 2:  $X_1, X_2, X_3$  and  $X_4$

		3	4				
	5		7				
4				6			
2	1						
		1		6			
				5		3	
					7	2	
							8

	6	3					
3			7				
4						1	
	1			7			
			2		6		
				5	4		
		5				2	
							8

	6			2			
3							6
		7					1
				7	3		
5			2				
				1		4	
	4	5					
							8

1				2			
			4		6		
		7			2		
					3	5	
5	7						
		6	1				
	4		3				
							8

1			4				
	5			4			
				6	2		
2						5	
	7	1					
		6					3
			3		7		
							8

1	6						
3				4			
					2	1	
					7		5
	7		2				
		6				4	
		5	3				
							8

			4	5			
	5	2					
	3			6			
2			6				
		1				4	
7							3
				1	7		
							8

1					5		
		2		4			
	3				2		
			6			5	
	7						4
7		6					
			3	1			
							8

		3			5		
		2	7				
4	3						
	1		6				
					6	4	
7				5			
				1		2	
							8

	6				5		
3		2					
	3					1	
			6	7			
			2			4	
7					4		
		5		1			
							8

			4	2			
	5						6
		7		6			
2					3		
5		1					
			1			3	
	4				7		
							8

Table 3:  $Y_1, \dots, Y_{12}$

8		7					
	3					1	
6			3				
				8			
			2				4
						5	2
				4	7		
	1	6					

Table 4:  $W_1$

We now examine properties of the main class representatives  $X$ ,  $Y$  and  $W$ .

$X$  and  $Y$  are in the class of Latin squares Meynert [15] refers to as *rcs - symmetric*. Thus,  $(i, j; k) \in L$  implies entry  $(j, k; i) \in L$  and entry  $(k, i; j) \in L$ . Although they are not presented in this way here, they appear in this class.

Each of the sets  $X_i$  and  $Y_i$  has  $11,662,776=2^3 \cdot 3^2 \cdot 161,983$  completions.  $W_1$  has  $7,075,188 = 2^2 \cdot 3^5 \cdot 29 \cdot 251$  completions.

Each of the sets  $X_i, Y_i$ , and  $W_i$  have  $R(j), C(j)$ , and  $E(j)$  for  $j = 1, \dots, 8$  equal to 1 or 2. Since the size of each set is 15, exactly one row and column contains one entry, and one element occurs once.

$X$  contains 21 intercalates,  $Y$  contains 7 intercalates, and  $W$  contains 9 intercalates. Note that  $X$  has three times as many intercalates as  $Y$  and  $Y$  has three times as many sets as  $X$  with the property above.

The sets  $Y_i$  are symmetric about the main diagonal.

For both  $X$  and  $Y$ , there are seven intercalates intersecting the entry  $(8, 8; 8)$  and in both cases the sets  $X_i$  and  $Y_i$  use in total exactly 43 entries from the squares  $X$  and  $Y$  respectively. In  $X_i$  each entry present occurs once, except  $(8, 8; 8)$  which occurs 4 times and a transversal (a set in which every element, row, and column occurs exactly once) in which each entry (except  $(8, 8; 8)$ ) occurs 3 times; in  $Y_i$  each entry present occurs 4 times, except  $(8, 8; 8)$  which occurs 12 times.

$X$  and  $Y$  seem to have a similar spectrum of possible critical set sizes - in both cases, the spectrum found so far is 21 to 28. This leads naturally to a question, given that they share so many other properties.

**Question.** Do there exist two main classes of Latin squares of the same order such that there is a mapping between critical sets from one main class to the other?

## 4 Ideas for future research

Further research could include looking at the 1707 main classes of order 9 which contain no intercalates, to check whether any critical sets of size less than  $\lfloor \frac{9^2}{4} \rfloor$



exist there. Alternatively, the computation could be repeated to verify “Conjecture 1” of Bate and van Rees [3] that the critical set of size  $\lfloor \frac{n^2}{4} \rfloor$  exists only in the main class  $\mathbb{Z}_n$ , for  $n = 8$ .

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## References

- [1] Adams, P. and Khodkar, A. Smallest critical sets for the Latin squares of orders six and seven. *J. Combin. Math. Combin. Comput.* 37 (2001), 225–237.
- [2] Bate, J. A. and van Rees, G. H. J. The size of the smallest strong critical set in a Latin square. *Ars Combin.* 53 (1999), 73–83.
- [3] Bate, J. A. and van Rees, G. H. J. Minimal and near-minimal critical sets in back-circulant Latin squares. *Australas. J. Combin.* 27 (2003), 47–61.
- [4] Bean, R. W. Critical sets in latin squares and related structures. Ph. D. Thesis, University of Queensland, 2001.
- [5] Cavenagh, N. Latin trade algorithms and the size of the smallest critical set in a latin square *Proc. Thirteenth Australasian Workshop on Combinatorial Algorithms*, (University of Queensland, Queensland, 2002), 61–74.
- [6] Curran, D. and van Rees, G. H. J. Critical sets in Latin squares. *Proceedings of the Eighth Manitoba Conference on Numerical Mathematics and Computing* (Univ. Manitoba, Winnipeg, Man., 1978), 165–168.
- [7] Donovan, D., Cooper, J., Nott, D. J., and Seberry, J. Latin squares: critical sets and their lower bounds. *Ars Combin.* 39 (1995), 33–48.
- [8] Donovan, D., Mahmoodian, E. S., Ramsay, C., and Street, A. P. Defining sets in combinatorics: a survey. *Surveys in Combinatorics 2003*, London Mathematical Society Lecture Note Series 307, Cambridge University Press, 115–174.
- [9] Hafer, L. BonsaiG. <http://www.cs.sfu.ca/~lou/BonsaiG/>
- [10] Horak, P., Aldred, R. E. L., and Fleischner, H. Completing latin squares: critical sets. *Journal of Combinatorial Designs* 10, no. 6 (2002), 419–432.
- [11] ILOG Inc. CPLEX Integer Programming Solver, Version 8.0. <http://www.cplex.com/>

- [12] Khodkar, A. On smallest critical sets for the elementary abelian 2-group. *Util. Math.* 54 (1998), 45–50.
- [13] Kolesova, G., Lam, C., and Thiel L. On the number of  $8 \times 8$  Latin squares. *J. Combin. Theory Ser. A* 54 (1990), no. 1, 143–148.
- [14] McKay, B. Latin squares data. <http://cs.anu.edu.au/~bdm/data/latin.html>
- [15] Meynert, A. Latin squares project. <http://www.cs.uvic.ca/~ameynert/lsq/>
- [16] Nelder, J. Critical sets in Latin squares. *CSIRO Division of Math. and Stats Newsletter* 38 (1977), 4.